Billiard systems with polynomial integrals of third and fourth degree

2001 J. Phys. A: Math. Gen. 34 2121

(http://iopscience.iop.org/0305-4470/34/11/305)

J. Phys. A: Math. Gen. **34** (2001) 2121–2124 www.iop.org/Journals/ja PII: S0305-4470(01)15476-4

Billiard systems with polynomial integrals of third and fourth degree

Tatiana Kozlova

Department of Mechanics and Mathematics, Moscow Lomonosov State University, Vorob'evy Gory, 119899 Moscow, Russia

Received 2 June 2000, in final form 9 January 2001

Abstract

The problem of the existence of polynomial-in-momenta first integrals for dynamical billiard systems is considered. Examples of billiards with irreducible integrals of third and fourth degree are constructed with the help of the integrable problems of Goryachev–Chaplygin and Kovalevsky from rigid body dynamics.

PACS numbers: 0210N, 4520J

1. Introduction

Consider a dynamical system with elastic impacts in a domain diffeomorphic to a disc. In what follows we will call this system a curved billiard. Let θ, φ mod 2π be the polar coordinates. We assume that the Lagrangian has the form

$$L = \tfrac{1}{2}(\dot{\theta}^2 + f(\theta)\dot{\varphi}^2) - V(\theta, \varphi) \tag{1.1}$$

where f is a regular positive function which in the neighbourhood of the point $\theta = 0$ has the asymptote $f = \kappa\theta^2 + \mathrm{o}(\theta^2)$, $\kappa > 0$. For example, $f = \theta^2$ in the case of a plane disc; $f = \sin^2\theta$ in the case of a spherical disc. One can imagine that a system with the Lagrangian (1.1) describes the motion of a particle with unit mass on a surface of revolution (homeomorphic to a disc) when potential forces with a potential energy V are applied. This function is 2π-periodic in φ. We are interested in the motion on the disc $0 \leqslant \theta \leqslant \theta_0$, $\theta_0 = \mathrm{const} > 0$.

One can make a transition to canonical variables θ, φ, p_θ, p_φ by the standard procedure. In these variables a Hamiltonian has the form

$$H = \tfrac{1}{2}(p_\theta^2 + g(\theta)p_\varphi^2) + V(\theta, \varphi) \tag{1.2}$$

where $g = 1/f$. After an elastic impact on the boundary $\theta = \theta_0$, generalized velocities transform as

$$\dot{\theta} \to -\dot{\theta} \qquad \dot{\varphi} \to \dot{\varphi}.$$

Canonical momenta transform in the same way:

$$p_\theta \to -p_\theta \qquad p_\varphi \to p_\varphi. \tag{1.3}$$

In [1] the problem of existence of polynomial-in-momenta first integrals, independent from the total energy H, in billiard systems of this type is investigated. It is shown that a linear integral only exists if φ is a cyclic coordinate; existence of a quadratic integral leads to the separation of variables

$$V = \frac{v(\varphi)}{f(\theta)} \tag{1.4}$$

where v is a 2π-periodic function.

The situation with integrals of higher degrees is more complicated. In [1] it is shown that if a third degree integral is an integral for a billiard in every disc $\theta \leqslant \alpha$, $\alpha = \text{const} > 0$, then it can be transformed to a linear integral (V does not depend on φ).

In this paper examples of billiards with irreducible polynomial integrals of third and fourth degrees are given. The integrable problems of Goryachev–Chaplygin and Kovalevsky from rigid body dynamics (see, e.g., [2]) are used to construct these systems. A similar method was used in [3] for the construction of geodesic flows on a sphere with integrals of third and fourth degrees (see also [4]).

The obtained integrals are integrals only for billiards in a certain domain $\theta \leqslant \alpha$. The dynamics of the constructed system is investigated by the Poincaré cross section method.

2. Third degree integral

Define

$$f = \frac{\sin^2\theta}{4\sin^2\theta + \cos^2\theta} \qquad V = -\tfrac{1}{8}\sin\theta\sin\varphi \tag{2.1}$$

and consider Hamilton's equations with the Hamiltonian (1.2) (where one should substitute f and V with formulae (2.1)) which admit the polynomial integral of third degree

$$\Phi = p_\varphi\left(\frac{p_\varphi^2\cos^2\theta}{\sin^2\theta} + p_\theta^2\right) + \left(\frac{p_\varphi\cos\theta\sin\varphi}{\sin\theta} - p_\theta\cos\varphi\right)\frac{\cos\theta}{2}. \tag{2.2}$$

This system can be obtained by Routh reduction in the problem on rotation of a heavy rigid body with a fixed point in the Goryachev–Chaplygin case (see, e.g., [4]). Variables θ, φ are the nutation angle and the precession angle; the angle of pure rotation is excluded as a consequence of the area integral which equals zero.

In order to obtain an integrable system with elastic impacts, one should restrict consideration of motion by the disc

$$0 \leqslant \theta \leqslant \alpha \qquad \alpha = \pi/2.$$

When $\theta = \pi/2$, function (2.2) does not change after substitution (1.3).

One can note that for other values of α the third degree polynomial (2.2) will not be an integral of a system with elastic impacts. The integral (2.2) cannot be reduced to an integral of lesser degree, because the potential energy does not have the form of (1.4).

To get an idea about the motions of the system described, we will use the Poincaré cross section method, that was also used by Birkhoff [5] to describe elastic billiards in convex domains. On the boundary of a disc (when $\theta = \pi/2$) the energy integral has the form

$$p_\theta^2 + 4p_\varphi^2 - \sin\varphi = h. \tag{2.3}$$

We assume that $h > 1$ (otherwise the region of a possible motion will be a part of a disc). When h is fixed, equation (2.3) can be rewritten in the following parametric form:

$$p_\theta = (h + \sin\varphi)^{1/2}\sin\psi \qquad p_\varphi = \tfrac{1}{2}(h + \sin\varphi)^{1/2}\cos\psi. \tag{2.4}$$

The angles φ mod 2π, $0 \leqslant \psi \leqslant \pi$ define a trajectory of energy h that intersects the boundary. Let φ_0, ψ_0 be the values of angles in an initial moment, and φ_1, ψ_1 the values of the first intersection with the boundary. The dynamics of a system with reflections is reduced to investigation of a mapping of a ring $K = \varphi \bmod 2\pi, 0 \leqslant \psi \leqslant \pi$:

$$\varphi_0, \psi_0 \to \varphi_1, \psi_1. \tag{2.5}$$

This mapping is integrable: an integral is the polynomial (2.2), where one should put $\theta = \pi/2$ and make a substitution (2.4). Up to an unessential constant multiplier this integral has the form

$$\Phi = (h + \sin\varphi)^{3/2} \sin^2\psi \cos\psi.$$

The level curves of the function Φ are invariant curves of the mapping (2.5). The system has a minimum, a maximum and two saddle points. The lines $\psi = 0$ and π consist of singularities.

3. Fourth degree integral

Let

$$f = \frac{\sin^2\theta}{2\sin^2\theta + \cos^2\theta} \qquad V = -\sin\theta\sin\varphi$$

in equation (1.1), in which case this system corresponds to the Kovalevsky case when the area integral equals zero. The Kovalevsky integral has the form

$$\Phi = (p_\theta^2 + p_\varphi^2 \operatorname{ctg}^2\theta)^2 + 4\sin\theta\sin\varphi(p_\varphi^2\operatorname{ctg}^2\theta - p_\theta^2) - 8p_\theta p_\varphi\cos\theta\cos\varphi + 4\sin^2\theta. \tag{3.1}$$

This polynomial of fourth degree is an integral of a billiard in the disc

$$0 \leqslant \theta \leqslant \pi/2$$

because when $\theta = \pi/2$, function (3.1) does not change after substitution (1.3).

On the boundary of the disc the energy integral has the form

$$p_\theta^2 + 2p_\varphi^2 - 2\sin\varphi = 2h.$$

Analogous to section 2, we put

$$p_\theta = \sqrt{2}(h + \sin\varphi)^{1/2}\sin\psi \qquad p_\varphi = (h + \sin\varphi)^{1/2}\cos\psi. \tag{3.2}$$

The formulae are correctly defined for $h > 1$. Substituting (3.2) in (3.1) and taking $\theta = \pi/2$, we obtain (up to an inessential constant multiplier)

$$\Phi = (h + \sin\varphi)\sin^2\psi(2(h + \sin\varphi)\sin^2\psi - 4\sin\varphi)$$

which is the integral of Poincaré mapping of the ring

$$\varphi \bmod 2\pi \qquad 0 \leqslant \psi \leqslant \pi$$

onto itself. The singularities of the system are two maximum points, two minimum points, two saddle points, and also the lines $\psi = 0$ and π.

4. Conclusion

Polynomial-in-momenta integrals play an important role when the equations of motion of dynamical systems are investigated. In this paper the structure and conditions for the existence of linear and quadratic integrals are studied extensively. The problem of polynomial irreducible integrals of degree $\geqslant 3$ is much more complicated and has still not been fully investigated. The topology of the configuration space also plays a vital part in the existence conditions of new integrals.

It is highly probable that there are no billiard systems on a plane with additional integrals of degree $\geqslant 3$ (for discussion of this hypothesis see [1]). However, as is shown in this paper, it is possible to construct examples of billiard systems with irreducible integrals of third and fourth degree on surfaces of revolution with a specially chosen boundary.

References

[1] Kozlova T V 1999 On polynomial integrals of systems with elastic impacts *Regular Chaotic Dyn.* **4** 83–90

[2] Whittaker E T 1927 *A Treatise on Analytical Dynamics* 3rd edn (Cambridge: Cambridge University Press)

[3] Kozlov V V and Denisova N V 1994 Polynomial integrals of geodesic flows on two-dimensional torus *Math. Sbornik.* **185** (12) 49–64 (in Russian)

[4] Bolsinov A V, Kozlov V V and Fomenko A T 1995 Maupertuis principle and geodesic flows on a sphere apperining from integrable cases of rigid body dynamics *UMN* **50** (3) 3–32 (in Russian)

[5] Birkhoff G D 1927 *Dynamical systems (Am. Math. Soc. Colloq. Publ. 9)* (New York: American Mathematical Society)